JOURNAL OF DIFFERENTIAL EQUATIONS 12, 330–352 (1972)

Isolated Invariant Sets in Compact Metric Spaces

RICHARD C. CHURCHILL

Department of Mathematics, Hunter College, City University of New York, New York 10021

Received October 21, 1971

INTRODUCTION

In the terminology of Ura and Kimura, an invariant set of a flow is *isolated* if it is the maximal invariant set in the interior of some neighborhood of itself [26, 27]. For smooth flows, Conley and Easton have characterized such an invariant set as being maximal in the interior of an "isolating block" [10, 11]. A special feature of such a neighborhood is that in terms of their Čech cohomology, particular subsets of the boundary can be related to the invariant set inside. In particular, conditions similar to those given by Ważewski [28, 29], described in [15], guarantee that this set is nonempty.

Isolating block techniques have been applied to several problems. Degenerate critical points [5, 20], the two-body problem [14], the restricted three-body problem [6, 16], the full three-body problem [17], minimal sets [25], and most recently shock-wave phenomena [21–24], have been studied in such a context. Additional results may be found in [7, 8, 13, 18, and 32].

This paper places the work of Conley and Easton on isolated invariant sets into the setting of a compact metric space. The notion of an "isolating block" is developed in this framework, and an invariant set is shown to be isolated if and only if it can be realized as the maximal invariant set in the interior of such a set. We proceed from this characterization to develop an algebraic invariant for an isolated invariant set, somewhat similar to the Morse index of an isolated critical point. Our index is shown to be stable under perturbation, where we consider the collection of flows on a particular space, and endow this collection with the compact-open topology. A second algebraic invariant for an isolated invariant set is also developed, this being a long exact sequence relating our index to the orbits asymptotic to the given set. In [9], Conley uses this sequence to generalize the Morse–Smale inequalities, and in [8, 18] Conley and Montgomery use the index and sequence to develop a sheaf-theoretic approach to a continuation and bifurcation theory for isolated invariant sets.

In Theorem 2.1 a preliminary characterization of isolated invariant sets is given, to be used eventually to construct isolating blocks. This theorem can be regarded as an extension of the work of Zubov [33], Bhatia [3], and Auslander and Siebert [1, 2] on generalized Liapunov functions and asymptotic stability. These authors show that an invariant set I is positively asymptotically stable if and only if there is a continuous real-valued function defined on some neighborhood U of I which is strictly decreasing along orbit segments in $U - I$, and is zero exactly on I. In analogy, we show that an invariant set S is isolated if and only if there is a continuous function with values in $[0, \infty]$, defined on a closed set N containing S in its interior, which is strictly increasing, strictly decreasing, of first strictly increasing and then strictly decreasing, on orbit segments in $N - S$, and is infinite exactly on S. In our concluding section of examples, we include several applications of isolating block techniques to problems concerning asymptotic stability.

A substantial portion of the results of this paper are taken from the author's doctoral thesis, written under the direction of Charles Conley. It is a pleasure to acknowledge his encouragement and assistance.

1. Basic Definitions and Notations

R will denote the reals, and X will always denote a compact metric space, with metric d. A *flow* on X is a continuous function $f : X \times R \to X$ such that (a) $f(p, 0) = p$ and (b) $f(p, s + t) = f(f(p, s), t)$ for all $p \in X$ and s, $t \in R$. Throughout this paper f will be a flow on X.

If $A \subset X$, $J \subset R$ we write $A \cdot J$ for $f(A \times J)$. With this notation (a) and (b) become (a′) $p \cdot 0 = p$ and (b′) $p \cdot (s + t) = (p \cdot s) \cdot t$, respectively.

For each $p \in X$, the sets $p \cdot R$, $p \cdot [0, \infty)$, and $p \cdot (-\infty, 0]$ are the *orbit*, *positive orbit*, and *negative orbit* of p, and are denoted $\mathcal{O}(p)$, $\mathcal{O}^+(p)$, and $\mathcal{O}^-(p)$, respectively. If $\mathcal{O}(p) = \{p\}$, then p is a *rest point* of f. A set $I \subset X$ is *invariant* (w.r.t. f) if $\mathcal{O}(p) \subset I$ for all $p \in I$. An equivalent definition of invariance is $I \cdot R = I$. The closure of an invariant set is again invariant; in particular, the closure of an orbit is invariant.

Next to a rest point, the simplest invariant set is a *periodic orbit*. This is an orbit $\mathcal{O}(p)$ containing more than one point, but such that $p \cdot t = p$ for some $t > 0$. The smallest such t is the *period* of $\mathcal{O}(p)$. It is well-known that a periodic orbit is homeomorphic to S^1.

If $p, q \in X$ and $p \cdot t_n \to q$ for some sequence $t_n \searrow -\infty$, then q is an *α-limit point* of $\mathcal{O}(p)$. If $t_n \nearrow \infty$, then q is an *ω-limit point* of $\mathcal{O}(p)$. $\mathcal{O}_\alpha(p)$ and $\mathcal{O}_\omega(p)$ are the *α- and ω-limit sets* of $\mathcal{O}(p)$, i.e., the respective collections of α- and ω-limit points of $\mathcal{O}(p)$. These are closed invariant sets.

For $p \in M \subset X$, $\mathcal{O}(p, M)$ will denote that component of $\mathcal{O}(p) \cap M$ which contains p. This is the *orbit segment* of p in M. Similarly, $\mathcal{O}^+(p, M)$ and $\mathcal{O}^-(p, M)$ are those components of $\mathcal{O}^+(p) \cap M$ and $\mathcal{O}^-(p) \cap M$ which contain p. These will usually be regarded as curves, parameterized by the time-variable t. If $A \subset M$, then $\mathcal{O}(A, M) = \bigcup \mathcal{O}(p, M)$, $\mathcal{O}^+(A, M) = \bigcup \mathcal{O}^+(p, M)$, and $\mathcal{O}^-(A, M) = \bigcup \mathcal{O}^-(p, M)$, where the unions are taken over all $p \in A$.

For a general reference for flows on metric spaces, see Chapter 5 of [19].

For any set $A \subset X$, $\mathrm{Cl}(A)$, $\mathrm{int}(A)$ and ∂A will denote the closure, interior, and boundary of A. If $A \subset M \subset X$, $H^q(M, A)$ will denote the q-th Čech cohomology group of the pair (M, A), $q \geqslant 0$, where the coefficients are taken in some fixed ring. If Y is any set and $h : X \to Y$ is any function, then $h \mid A$ will denote the restriction of h to A.

2. Isolated Invariant Sets

An invariant set S is *isolated* (w.r.t. f) if it is the maximal invariant set in some neighborhood of itself; in this paper S will always be such a set. Observe that S must be the maximal invariant set in the interior of an *isolating neighborhood*, i.e., a closed set N such that $p \in \partial N$ implies $\mathcal{O}(p) \not\subset N$. Such an N is said to *isolate* S. Note that if $M \subset N$ is closed and $S \subset \mathrm{int}(M)$, then M also isolates S.

For the remainder of the section we let N isolate S.

Let $\rho : N \to [0, \infty)$ be a continuous function such that $\rho(p) = 0$ if and only if $p \in S$; if $S \neq \varnothing$ we could let $\rho(p) = d(p, S) = \inf\{d(p, q) \mid q \in S\}$. Define $l^\pm : N \to [0, \infty)$ by

$$l^+(p) = \inf\{\rho(q) \mid q \in \mathcal{O}^+(p, N)\}, \qquad l^-(p) = \inf\{\rho(q) \mid q \in \mathcal{O}^-(p, N)\}.$$

For $p \in N$, $l^+ \mid \mathcal{O}(p, N)$ is nondecreasing (as a function of t), and $l^- \mid \mathcal{O}(p, N)$ is nonincreasing.

For all $\epsilon > 0$, define $V_\epsilon = \rho^{-1}([0, \epsilon))$.

Lemma 2.1. *$l^+ : N \to [0, \infty)$ is lower semicontinuous. Further, if $p \in V_\epsilon$ is a discontinuity point of this map, then there is a $p_\epsilon \in \partial N$ such that $\mathcal{O}^+(p_\epsilon, N) \cap V_\epsilon \neq \varnothing \neq \mathcal{O}^-(p_\epsilon, N) \cap V_\epsilon$.*

Proof. Choose $p \in N$. If $\mathcal{O}^+(p) \subset N$, then $\mathcal{O}_\omega(p) \subset S$, since $\mathcal{O}_\omega(p)$ is invariant. Hence $l^+(p) = 0$. *If $q \in N$ is close to p, then $\mathcal{O}^+(q)$ passes close to S, showing that $l^+(q)$ is small. Thus l^+ is continuous at p.*

If $\mathcal{O}^+(p) \not\subset N$, then choose $t_0 > 0$ such that $p \cdot t_0 \notin N$. If $p_n \to p$, we may assume that $p_n \cdot t_0 \notin N$ for all n. If U is an open neighborhood of $p \cdot [0, t_0]$, then $\mathcal{O}^+(p_n, N) \subset U$ for large n, showing that $\liminf_{n\to\infty} l^+(p_n) \in \mathrm{Cl}(\rho(U \cap N))$.

But U is arbitrary, so that actually $\liminf_{n\to\infty} l^+(p_n) \in \mathrm{Cl}(\rho(\mathcal{O}^+(p, N)))$, and since $l^+(p)$ is the infimum of this last set, we have shown the lower semicontinuity of l^+ at p.

If $p \in V_\epsilon$ is a discontinuity point of l^+, then there is a sequence $p_n \to p$ such that $l^+(p_n) \to l^+(p) + \delta$, where $\delta > 0$. But then $l^-(p) + \delta = \lim l^+(p_n) \leqslant \lim \rho(p_n) = \rho(p)$, showing that $l^+(p) < \rho(p) - \delta/2$. By the definition of l^+ we conclude that there is a $t_0 > 0$ such that $p \cdot [0, t_0] \subset N$ and $\rho(p \cdot t_0) < \rho(p) - \delta/2$, and in fact we can assume that $\rho(q) < \rho(p) - \delta/2$ for all q in some open $U \subset N$ containing $p \cdot t_0$. Since $p_n \cdot t_0 \in U$ for large n, and since $l^+(p_n) > l^+(p) + \delta/2$, there must be a $t_n \in [0, t_0]$ such that $p_n \cdot t_n \in \partial N$. By choosing a subsequence we may assume that $t_n \to t' \in [0, t_0]$, so that $p_n \cdot t_n \to p \cdot t' \in \partial N$. Setting $p_\epsilon = p \cdot t'$, the proof is complete. Q.E.D.

LEMMA 2.2. *There is an open $V \subset N$ containing S on which l^+ and l^- are continuous.*

Proof. Otherwise we may assume that there is a discontinuity point of l^+ in each neighborhood $V_{1/n}$ of S, $n = 1, 2,\ldots$. Choosing $p_{1/n}$ as in Lemma 2.1 and letting $p' \in \partial N$ be a limit point of this sequence, it is easy to verify that $\mathcal{O}(p') \subset N$, and this contradicts the fact that N is an isolating neighborhood. Q.E.D.

Define $l = \max\{l^+, l^-\}$, so $l : V \to [0, \infty)$ is continuous, and note that $S = l^{-1}(\{0\})$. Let $c > 0$ be such that $M = l^{-1}([0, c]) \subset V$, and define $l' : X \to [0, \infty)$ by $l'(p) = l(p)$ if $p \in M$; $l'(p) = c$, otherwise. Further, if $p \in M$, set

$$L^+(p) = \int_{\mathcal{O}^-(p,M)} \{c - l'(p \cdot s)\}\, ds, \qquad L^-(p) = \int_{\mathcal{O}^+(p,M)} \{c - l'(p \cdot s)\}\, ds.$$

If $p \in M$ and $\mathcal{O}^-(p) \not\subset M$, then $p \cdot t_0 \in V - M$ and $p \cdot [t_0, 0] \subset V$ for some $t_0 < 0$. If $p_n \to p$, then for all n we may assume these same inclusions for $p_n \cdot t_0$ and $p_n \cdot [t_0, 0]$. Next observe that $q \cdot [t_0, 0] \cap M = \mathcal{O}^-(q, M)$ for $q = p$ or p_n, since by the convex nature of l restricted to orbit segments, once an orbit leaves M in negative time it cannot return to M without first leaving V. Finally, note that $c - l'(q \cdot s) = 0$ if $q \cdot s \notin M$, so that $L^-(q) = \int_{t_0}^0 \{c - l'(q \cdot s)\}\, ds$ for $q = p$ or p_n. Thus $L^+(p_n) \to L^+(p)$ by the dominated convergence theorem. If $\mathcal{O}^-(p) \subset M$, then $L^+(p) = \infty$, and if $p_n \to p$ it is not difficult to see that $\liminf_{n\to\infty} L^+(p_n) = \infty$, so that again $L^+(p_n) \to L^+(p)$. Thus $L^+ : M \to [0, \infty]$, and similarly $L^- : M \to [0, \infty]$, is a continuous function. Here $[0, \infty]$ is the space obtained from $[0, \infty)$ by adjoining an ideal point ∞ and topologizing the resulting set with the order topology. Spaces $[-\infty, 0]$ and $[-\infty, \infty]$ are constructed analogously, the latter by adjoining two ideal points. $[0, \infty]$, $[-\infty, 0]$ and $[-\infty, \infty]$ are each homeomorphic to the closed unit interval.

THEOREM 2.1. *An invariant set S is isolated if and only if there is a closed set M with $S \subset \operatorname{int}(M)$ and a continuous function $L : M \to [0, \infty]$ with the following properties:*

(a) $S = L^{-1}(\{\infty\})$;

(b) $\partial M = L^{-1}(\{0\})$; *and*

(c) $L = \min\{L^+, L^-\}$, *where $L^\pm : M \to [0, \infty]$ are continuous and such that for $p \in M - S$ we have $L^+ \mid \mathcal{O}(p, M)$ strictly increasing with t unless $\mathcal{O}_\alpha(p) \subset S$, in which case $L^+(p) = \infty$; and $L^- \mid \mathcal{O}(p, M)$ strictly decreasing with t unless $\mathcal{O}_\omega(p) \subset S$, in which case $L^-(p) = \infty$.*

Further, if S is isolated and N isolates S, then we may choose $M \subset \operatorname{int}(N)$.

Proof. If S is isolated choose M, L^+ and L^- as in the preceding two paragraphs, and let $L = \min\{L^+, L^-\}$.

For the converse, assume $S \subset \operatorname{int}(M)$, and that $L : M \to [0, \infty]$ is a continuous function satisfying (a)–(c). If $c \in [1, \infty)$ set $M_c = L^{-1}([c, \infty]) \subset \operatorname{int}(M)$, and observe that $\bigcap M_c = S$. If $p \in M_c$, then $p \in M - S$, and from (c) we find that $\mathcal{O}(p) \not\subset M_c$. It is then immediate that S is the maximal invariant set in M_1. Q.E.D.

3. ISOLATING BLOCKS

If $\Sigma \subset X$ and $\delta > 0$, define a map $\varphi_\delta : \Sigma \times (-\delta, \delta) \to X$ by setting $\varphi_\delta(p, t) = p \cdot t, p \in \Sigma, t \in (-\delta, \delta)$. If φ_δ is a homeomorphism, then $\Sigma \cdot (-\delta, \delta)$ is a *collar* of Σ (w.r.t. f). If this is the case, then we see that Σ is a strong deformation retract of $\Sigma \cdot (-\delta, \delta)$. If φ_δ is a homeomorphism with open range, then Σ is a *local section* (w.r.t. f).

For any local section $\Sigma \subset X$, define $\sigma : X \to [0, \infty]$ by

$$\sigma(p) = \sup\{t \geqslant 0 \mid p \cdot [0, t] \cap \Sigma = \varnothing\} \qquad \text{if} \quad p \in X - \Sigma,$$

$\sigma(p) = 0$, if $p \in \Sigma$.

PROPOSITION 3.1. *Let Σ be a local section with collar $\Sigma \cdot (-\delta, \delta)$. Then $\sigma \mid X - \Sigma \cdot (0, \delta)$ is upper semicontinuous. If $p \in X$ is a discontinuity point of this map, then there is a $t \in (0, \sigma(p))$ such that $p \cdot t \in \operatorname{Cl}(\Sigma) - \Sigma$.*

Proof. $\sigma \mid \Sigma \cdot (-\delta, 0]$ is the composition of a homeomorphism and a projection, and is thus continuous. If $p \in X - \Sigma \cdot (-\delta, \delta)$, then upper semicontinuity follows trivially if $\sigma(p) = \infty$. If $\sigma(p) < \infty$, then $p \cdot \sigma(p) \in \Sigma$, and thus for $0 < \epsilon < \delta$ we have $p \cdot (\sigma(p) + \epsilon) \subset \Sigma \cdot (0, \delta)$. If $p_n \to p$, then for large n we have $p_n \cdot (\sigma(p) + \epsilon) \subset \Sigma \cdot (0, \delta)$, since this last set is open. But then $p_n \cdot [0, \sigma(p) + \epsilon] \cap \Sigma \neq \varnothing$ for such n, showing that $\sigma(p_n) \leqslant \sigma(p) + \epsilon$, and upper semicontinuity at p follows.

If $p \in X - \Sigma \cdot (-\delta, \delta)$ is a point of discontinuity, then $\lim_{n \to \infty} \sigma(p_n) = t < \sigma(p)$ for some sequence $p_n \to p$, and thus $p_n \cdot \sigma(p_n) \to p \cdot t \in \mathrm{Cl}(\Sigma) - \Sigma$. Q.E.D.

An analogous result holds for $\sigma' : X \to [-\infty, 0]$, defined by

$$\sigma'(p) = \inf\{t \leqslant 0 \mid p \cdot [t, 0] \cap \Sigma = \varnothing\} \qquad \text{if} \quad p \in X - \Sigma,$$

and $\sigma'(p) = 0$ if $p \in \Sigma$.

We need a sufficient condition for the existence of local sections. Let $V \subset X$ be open, and let $g : V \to R$ be continuous. f is *gradient-like* on V w.r.t. g if for each $p \in V$ we have $g(p) > g(p \cdot t)$ whenever $p \cdot [0, t] \subset V$, $t > 0$, and $p \neq p \cdot t$.

PROPOSITION 3.2. *Let V and $g : V \to R$ be as above. If $c \in R$ is constant and $\Sigma \subset g^{-1}(\{c\})$ is such that*

(a) $\mathrm{Cl}(\Sigma) \subset V$,
(b) $\mathrm{Cl}(\Sigma)$ *contains no rest points, and*
(c) Σ *is relatively open in* $g^{-1}(\{c\})$,

then Σ is a local section. Further, if $\delta > 0$ is such that $\mathrm{Cl}(\Sigma) \cdot [-\delta, \delta] \subset V$, then $\Sigma \cdot (-\delta, \delta)$ is a collar of Σ.

Proof. Notice by (a) that there must be a $\delta > 0$ such that $\mathrm{Cl}(\Sigma) \cdot [-\delta, \delta] \subset V$. For any such $\delta > 0$ let $\varphi_\delta : \mathrm{Cl}(\Sigma) \times [-\delta, \delta] \to X$ be defined by $\varphi_\delta(p, t) = p \cdot t$, where $p \in \Sigma$, $t \in [-\delta, \delta]$. Because f is gradient-like in V and (b) holds, φ_δ must be one-one. But then φ_δ is a homeomorphism onto its range, for it is certainly continuous, and its domain and range are compact metric spaces.

It remains to show that $\Sigma \cdot (-\delta, \delta)$ is open. If p is in this set, then there is a $q \in \Sigma$ and a $t_0 \in (-\delta, \delta)$ such that $p = q \cdot t_0$. Let $J \subset \mathrm{Cl}(J) \subset (-\delta, \delta)$ be an open interval containing t_0 and 0, and observe that $g \mid p \cdot J$, considered as a function on J, must decrease from above c to below c as t increases. If $p_n \to p$ then $p_n \cdot J \to p \cdot J$ and $g \mid p_n \cdot J \to g \mid p \cdot J$. But then for large n $p_n \cdot J \cap g^{-1}(\{c\})$ must contain a point q_n, and we see that $q_n \to q$. Because Σ is relatively open in $g^{-1}(\{c\})$, these q_n must eventually be in Σ, and hence $p_n \in q_n \cdot (-\delta, \delta)$ must eventually be in $\Sigma \cdot (-\delta, \delta)$. This shows that $\Sigma \cdot (-\delta, \delta)$ is open, and completes the proof. Q.E.D.

If $p \in X$ is not a rest point of f, choose $t_0 \in R$ such that $p \cdot t_0 \neq p$, and for $q \in X$ define $g(q) = \int_0^{t_0} d(q \cdot s, p)\, ds$. Considered as a function of t, $g(p \cdot t)$ has derivative $d(p \cdot t_0, p) > 0$ at $t = 0$, and we conclude that if q is within some small neighborhood U of p, then $(\partial/\partial t)\, g(q \cdot t)\,|_{t=0} > 0$. Since g is obviously continuous, we have shown that f is gradient-like on U w.r.t. g, and it follows easily from Proposition 3.2 that there is a local section (w.r.t. f) containing p. This is a variation of a theorem of Whitney and Bebutov [30; or 19, Theorem 2.14, p. 333].

Definition 3.3. Let $B \subset X$ be a closed set, and let Σ^+, Σ^- be local sections with disjoint closures. Let $\delta > 0$ be such that $\Sigma^+ \cdot (-\delta, \delta)$ and $\Sigma^- \cdot (-\delta, \delta)$ are disjoint collars of Σ^+, Σ^-. B is an *isolating block*, or simply a *block*, if

(a) $(\mathrm{Cl}(\Sigma^\pm) - \Sigma^\pm) \cap B = \varnothing$;

(b) $\Sigma^+ \cdot (-\delta, \delta) \cap B = (\Sigma^+ \cap B) \cdot [0, \delta)$ and $\Sigma^- \cdot (-\delta, \delta) \cap B = (\Sigma^- \cap B) \cdot (-\delta, 0]$; and

(c) for each $p \in (\partial B - (\Sigma^+ \cup \Sigma^-))$ there are real numbers ϵ_1, ϵ_2, with $\epsilon_1 < 0 < \epsilon_2$, such that $p \cdot \epsilon_1 \in \Sigma^+$, $p \cdot \epsilon_2 \in \Sigma^-$, and $p \cdot [\epsilon_1, \epsilon_2] \subset \partial B$.

The requirement that $\mathrm{Cl}(\Sigma^+) \cap \mathrm{Cl}(\Sigma^-) = \varnothing$ is not essential, but for the blocks which will concern us it can always be arranged, and it simplifies many of the proofs.

In this paper the letter B will always denote an isolating block. Further, b^+ and b^- will always denote the sets $\Sigma^+ \cap \partial B$ and $\Sigma^- \cap \partial B$, respectively, and the letter b will be reserved to represent $\partial B - (b^+ \cup b^-)$. We see from (a) that b^+ and b^- are closed.

As an example of a block, take the unit square in the plane, i.e., $B = \{(x, y) \mid 0 \leqslant x, y \leqslant 1\}$, and let f be the flow generated by the equations $\dot{x} = 0$, $\dot{y} = -1$. Here we have

$$b^+ = \{(x, y) \in B \mid y = 1\}, \qquad \text{and} \qquad b^- = \{(x, y) \in B \mid y = 0\}.$$

Notice that a block B is an isolating neighborhood, so that the maximal invariant set $S = S(B)$ in the interior of B is an isolated invariant set. We now show that any isolated invariant set can be realized in this way.

Theorem 3.4. *An invariant set S is isolated if and only if it is the maximal invariant set in the interior of some isolating block B, i.e., if and only if $S = S(B)$ for some block B. Further, any isolating neighborhood of S contains such a B.*

Proof. Given an isolating neighborhood N of S, choose M and

$L : M \to [0, \infty]$ as in Theorem 2.1. Fix any $\theta \in (0, \infty)$, and define $C = L^{-1}((\theta, \infty])$,

$$D = \{p \in M \mid \ L \mid \mathcal{O}(p, M) \leqslant 3\theta\}, \quad \text{and} \quad E = \{p \in M \mid \ L \mid \mathcal{O}(p, M) \leqslant 2\theta\}.$$

Next let $\Sigma = \partial C - E$, and $\Sigma^{\pm} = \{p \in \Sigma \mid L(p) = L^{\pm}(p)\}$. $\Sigma^{\pm}$ are local sections by Proposition 3.2, and are easily seen to have disjoint closures. $\mathrm{Cl}(C - D) \subset N$ is a block having $\Sigma^{\pm}$ as the associated local sections, and obviously $S = S(B)$. Q.E.D.

Notice in the proof that $L^{-1}((4\theta, \infty]) \subset B$. It follows that we can make ∂B arbitrarily close to ∂M by choosing $\theta > 0$ small.

We will concentrate on one block B for the remainder of this section.

Define $\sigma^+(p) = \inf\{t \leqslant 0 \mid p \cdot [t, 0] \cap \Sigma^+ = \varnothing\}$ for $p \in B - \Sigma^+$, $\sigma^+(p) = 0$ if $p \in \Sigma^+$. Similarly, define $\sigma^-(p) = \sup\{t \geqslant 0 \mid p \cdot [0, t] \cap \Sigma^- = \varnothing\}$ if $p \in B - \Sigma^-$, $\sigma^-(p) = 0$ if $p \in \Sigma^-$. Then

$$\sigma^+ : B \to [-\infty, 0] \qquad \text{and} \qquad \sigma^- : B \to [0, \infty],$$

and it follows from Proposition 3.1 and the comment following that proof that these functions are continuous. σ^+ and σ^- are quite useful in the study of certain subsets of B.

PROPOSITION 3.5. *Let v, w be any subsets of $b^{\pm}$, with $v \subset w$. Then w is a strong deformation retract of $\mathcal{O}(v, B) \cup w$.*

Proof. The strong deformation retraction is

$$r^{\pm} : (\mathcal{O}(v, B) \cup w) \times [0, 1] \to \mathcal{O}(v, B) \cup w,$$

defined for $p \in \mathcal{O}(v, B) \cup w$ and $t \in [0, 1]$ by $r^{\pm}(p, t) = p \cdot (\sigma^{\pm}(p)t)$. Q.E.D.

Set $A^{\pm} = \{p \in B \mid \mathcal{O}^{\pm}(p) \subset B\}$, $A = A^+ \cup A^-$, and $a^{\pm} = A^{\pm} \cap b^{\pm}$. Notice that $S = A^+ \cap A^-$. As a corollary of Proposition 3.5 we see that $b^+ - a^+$ and $b^- - a^-$ are strong deformation retracts of $B - A$. Actually there is a much stronger relationship between these two sets.

PROPOSITION 3.6. *$b^+ - a^+$ and $b^- - a^-$ are homeomorphic.*

Proof. If $p \in (b^{\pm} - a^{\pm})$ set $\pi^{\pm}(p) = p \cdot \sigma^{\mp}(p)$. The continuous maps $\pi^{\pm} : b^{\mp} - a^{\mp} \to b^{\pm} - a^{\pm}$ are inverses of each other. Q.E.D.

Here are some additional properties of $A^{\pm}$, A and $a^{\pm}$ which are of interest.

PROPOSITION 3.7. (a) *$A^{\pm}$, A and $a^{\pm}$ are closed;*

(b) *$b^{\pm}$ is a strong deformation retract of $B - A^{\mp}$;*

(c) *if $p \in \partial b^{\pm}(\text{rel } \Sigma^{\pm})$, then $\mathcal{O}(p, B) = \mathcal{O}(q, \partial B)$ for some $q \in B$; and*

(d) *$a^{\pm} \subset \text{int}(b^{\pm})(\text{rel } \Sigma^{\pm})$.*

Proof. (a) follows since $A^{\pm} = (\sigma^{\mp})^{-1}(\{\pm\infty\})$, and (b) is immediate from Proposition 3.5. (c) and (d) will only be shown for Σ^+, the other case being completely analogous.

If $p \in \partial b^+(\text{rel } \Sigma^+)$, then there is a sequence $p_n \in \Sigma^+ - b^+$ such that $p_n \to p$. By Definition 3.3(b) we have $p_n \cdot \delta/2 \notin B$, but also $p \cdot \delta/2 \in (B - \Sigma^+)$. Thus $p \cdot \delta/2 \in \partial B$. Since $\Sigma^+ \cdot (-\delta, \delta) \cap \Sigma^- \cdot (-\delta, \delta) = \varnothing$, we must actually have $p \cdot \delta/2 \in b$. (c) now follows from Definition 3.3(c).

If $p \in a^+$ then $\mathcal{O}^+(p) \subset B$. But (c) and Definition 3.3(c) imply that for $p \in \partial b^+$ we must have $\mathcal{O}^+(p) \not\subset B$. This easily implies (d). Q.E.D.

Because $S = S(B) = A^+ \cap A^-$, it follows from Proposition 3.7(b) that if b^+ or b^- is not a strong deformation retract of B, then $S \neq \varnothing$.

4. THE INDEX

In this section we will need to deal with several insolating blocks. We will distinguish between their various subsets by subscripting, e.g., A_j^- will be the A^- of the block B_j. We will concentrate on a particular isolated invariant set S, and all blocks will isolate S.

LEMMA 4.1. *Let $B_2 \subset B_1$ be blocks. Then there is a strong deformation retraction h_{12} of A_1 onto A_2 such that $h_{12} \mid a_1^{\pm} \times \{1\} : a_1^{\pm} \to a_2^{\pm}$ is a homeomorphism. If $a_2^{\pm} \subset a_1^{\pm}$, then h_{12} is the identity map for all t.*

Proof. By pushing along the orbits of f we can define a strong deformation retraction of $A_1^{\pm} - (A_2^{\pm} \cap \text{int}(B_2))$ onto $a_2^{\pm}$. Since these two sets are disjoint we can combine these retractions into one, and by extending this map to be the identity on $A_3 \times [0, 1]$ we have a retraction as desired. Q.E.D.

For the remainder of the paper $H^q(Y, Z)$ and $H^*(Y, Z)$ will denote the q-th Čech cohomology group and the Čech cohomology algebra, respectively, of the topological pair (Y, Z), with coefficients in some fixed ring.

LEMMA 4.2. *For any pair of blocks B_i, B_j isolating S there is an isomorphism $\gamma_{ij} : H^*(A_i, a_i^+) \to H^*(A_j, a_j^+)$ such that γ_{ii} is the identity map and $\gamma_{jk} \circ \gamma_{ij} = \gamma_{ik}$. Further, if $(A_j, a_j^+) \subset (A_i, a_i^+)$, then γ_{ij} is the map induced by this inclusion.*

Proof. $B_i \cap B_j$ isolates S, and by Theorem 3.4 there is a block B within this set which also isolates S. By Lemma 4.1 we see that for $n = i, j$ there are strong deformation retractions h_n of A_n onto A which induce isomorphisms $r_n^* : H^*(A, a^+) \to H^*(A_n, a_n^+)$. We define $\gamma_{ij} = r_j^* \circ (r_i^*)^{-1}$, and verify by elementary diagram chasing that this map is actually independent of B. The various properties required of the maps γ_{ij} are easily shown. Q.E.D.

LEMMA 4.3. *Let B isolate S, and let $i : (A, a^+) \to (B, b^+)$ be inclusion. Then $i^* : H^*(B, b^+) \to H^*(A, a^+)$ is an isomorphism.*

Proof. For $p \in B$ let $\sigma(p) = \min\{\sigma^-(p), |\sigma^+(p)|\}$, so $\sigma : B \to [0, \infty]$ is continuous. Let $\theta > 0$ be an upper bound for $\sigma \mid \partial B$, and for $n = 1, 2,\ldots$ set $D_n = \{p \in B \mid \sigma \mid \mathcal{O}(p, B) < n\theta\}$, $d_n = D_n \cap b^+$. Notice that

$$D_n \subset \operatorname{int}(\operatorname{Cl}(D_{n+1})) \qquad \text{for all} \quad n.$$

From Proposition 3.5 and excision we obtain

$$\begin{aligned} H^*(B, b^+) &\approx H^*(B, b \cup \operatorname{Cl}(D_{n+1})) \\ &\approx H^*(B - D_n, (b^+ \cup \operatorname{Cl}(D_{n+1})) - D_n) \\ &\approx H^*(B - D_n, b^+ - d_n). \end{aligned}$$

But $(A, a^-) = \bigcap_{n=1}^{\infty} (B - D_n, b^+ - d_n)$, and so the result follows from the continuity of the Čech theory. Q.E.D.

Combining Lemmas 4.2 and 4.3 we have the following:

THEOREM 4.4. *For any pair of blocks B_i, B_j isolating S there is an isomorphism $\theta_{ij} : H^*(B_i, b_i^+) \to H^*(B_j, b_j^+)$ such that θ_{ii} is the identity map and $\theta_{jk} \circ \theta_{ij} = \theta_{ik}$. If $(B_j, b_j^+) \subset (B_i, b_i^+)$, then θ_{ij} is the map induced by this inclusion.*

COROLLARY 4.5. *The cohomology algebra $H^*(B, b^+)$ is an invariant of S, i.e., it is independent of the block B used to compute it, so long as B isolates S.*

$H^*(B, b^+)$ is the *index* of S.

PROPOSITION 4.6. *If B isolates S then the inclusion $i : S \subset A$ induces an isomorphism $i^* : H^*(A) \to H^*(S)$.*

Proof. Choose any neighborhood $U \subset \operatorname{int}(B)$ of S. From the final statement of Theorem 3.4 there is a block $B_1 \subset U$ which isolates S. From Lemma 4.1 we know that A_1 is a strong deformation retract of A. Thus, in any neighborhood of S there is a strong deformation retract of A which contains S, and our result now follows from the continuity of the Čech theory. Q.E.D.

Let B_1 and B_2 isolate S, and, using Theorem 3.4, choose a block $B_3 \subset B_1 \cap B_2$ which also isolates S. By Lemma 4.1 we see that the long exact sequences of (A_1, a_1^+) and (A_2, a_2^+) are each isomorphic to that of (A_3, a_3^+), and hence to each other. Thus the long exact sequence of (A_1, a_1^+) is another invariant of S. By Lemma 4.3 we can replace $H^q(A_1, a_1^+)$ by $H^q(B_1, b_1^+)$ in this sequence, $q \geqslant 0$, and by Proposition 4.6 we can replace $H^q(A)$ by $H^q(S)$, $q \geqslant 0$. We have obtained the following result.

THEOREM 4.7. *The long exact sequence*

$$\cdots \to H^q(B, b^+) \to H^q(S) \to H^q(a^+) \to H^{q+1}(B, b^+) \to \cdots$$

is an invariant of S, i.e., it is independent of the block B used to compute it, so long as B isolates S.

$\cdots \to H^q(B, b^+) \to H^q(S) \to H^q(a^+) \to H^{q+1}(B, b^+) \to \cdots$ is the *sequence* of S.

By fixing a block B isolating S, and using Proposition 4.6, we can use the long exact sequences of (B, b^+) and (A, a^+) to make a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(B, b^+) \arrow[r] \arrow[d, "i^*"] & H^q(B) \arrow[r] \arrow[d, "j_1^*"] & H^q(b^+) \arrow[r] \arrow[d, "j_2^*"] & H^{q+1}(B, b^+) \arrow[r] \arrow[d, "i^*"] & \cdots \\
\cdots \arrow[r] & H^q(A, a^+) \arrow[r] & H^q(S) \arrow[r] & H^q(a^+) \arrow[r] & H^{q+1}(A, a^+) \arrow[r] & \cdots
\end{tikzcd}$$

which we call the *diagram* of B. It is not an invariant of S, but it nevertheless proves quite useful in applications. The vertical maps are all induced by inclusions, and i^* is an isomorphism by Lemma 4.3.

5. STABILITY

Again we will deal with several isolating blocks, and again we will distinguish the various subsets by subscripting. But we will also deal with blocks in a flow f' different from f, and these will be distinguished by *underlinings*. If $A \subset X$ and $J \subset R$, then we will write $A : J$ for $f'(A \times J)$; in particular, $f'(p, t)$ will be written $p : t$. F will be the space of flows on X, endowed with the compact-open topology.

This section is for the purpose of proving:

THEOREM 5.1. *Let B be any block, let $V^{\pm}$ be disjoint collars of* $\operatorname{int}(b^{\pm})$ (rel $\Sigma^{\pm}$), *and let $V = V^{+} \cup V^{-}$. Then there is an open neighborhood $\Phi \subset F$ of f such that any $f' \in \Phi$ admits a block $\mathbf{B}$ such that $(\mathbf{B}, \mathbf{b}^{+}) \subset (B \cup V, b^{+} \cup V^{+})$, and such that this inclusion induces an isomorphism $H^{*}(B, b^{+}) \approx H^{*}(\mathbf{B}, \mathbf{b}^{+})$.*

It follows from Theorem 4.4 that if $\mathbf{B}_1$ is any block isolating $\mathbf{S} = \mathbf{S}(\mathbf{B})$, then $H^{*}(B, b^{+}) \approx H^{*}(\mathbf{B}_1, \mathbf{b}_1^{+})$. In this sense our index is stable under perturbation.

A local section w.r.t. f is not necessarily a local section w.r.t. nearby flows. Herein lies the major difficulty in proving Theorem 5.1.

LEMMA 5.2. *Let $U, V \subset X$ be open, with $\mathrm{Cl}(U) \subset V$, suppose that f has no rest points in V, and suppose that f is gradient-like on V* w.r.t. *a continuous function $g : V \to R$. Then for any $\epsilon > 0$ there is an open neighborhood $\Phi \subset F$ of f such that each $f' \in \Phi$ is gradient-like on U* w.r.t. *a continuous function $g' : U \to R$ with the property that $|g(p) - g'(p)| < \epsilon$ for all $p \in U$.*

Proof. Let $W \subset \mathrm{Cl}(W) \subset V$ be open and such that $\mathrm{Cl}(U) \subset W$, and let $\delta, \lambda \in (0, \epsilon/2)$ be such that $\mathrm{Cl}(U) \cdot [-\lambda, \lambda] \subset W$, $\mathrm{Cl}(W) \cdot [-\lambda, \lambda] \subset V$, and

$$\delta < g(q) - g(q \cdot \lambda) < \epsilon/2, \qquad q \in \mathrm{Cl}(W). \tag{$'$}$$

For Φ take a neighborhood of f in F such that for each $f' \in \Phi$, $\mathrm{Cl}(U) : [-\lambda, \lambda] \subset W$ and

$$|g(p \cdot s) - g(p : s)| < \delta, \qquad p \in \mathrm{Cl}(W), \qquad |s| \leqslant \lambda. \tag{$''$}$$

Now choose $f' \in \Phi$ and for $p \in U$ let $g'(p) = 1/(2\lambda) \int_{-\lambda}^{\lambda} g(p : s)\, ds$. We claim that this g' satisfies the conditions of the lemma. First, we compute that $\partial/(\partial t)\, g'(p : t)|_{t=0} = 1/(2\lambda)(g(p : \lambda) - g(p : -\lambda))$. By ($'$) we have $g(p \cdot \lambda) - g(p \cdot -\lambda) > 2\delta$, and coupled with ($''$) this shows that the derivative in question is negative. Hence f' is gradient-like on U w.r.t. g'. Next, again using ($'$) and ($''$), we have

$$\begin{aligned} |g'(p) - g(p)| &\leqslant \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} |g(p : s) - g(p)|\, ds \\ &\leqslant \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} |g(p : s) - g(p \cdot s)|\, ds + \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} |g(p \cdot s) - g(p)|\, ds \\ &\leqslant \frac{1}{2\lambda} \cdot 2\lambda \cdot \delta + \frac{1}{2\lambda} \cdot 2\lambda \cdot \epsilon/2 < \epsilon, \end{aligned}$$

and this completes the proof. Q.E.D.

LEMMA 5.3. *Let Σ be a local section* w.r.t. f, *and let* K_1, $K_2 \subset \Sigma$ *be compact and such that* $K_2 \subset \mathrm{int}(K_1)(\mathrm{rel}\ \Sigma)$. *Also, let* V *be any collar of* Σ. *Then there is an open neighborhood* Φ *of* f *in* F *such that each* $f' \in \Phi$ *admits two local sections* $\mathbf{\Sigma}_2 \subset \mathbf{\Sigma}_1$ *with the following property: There are positive constants* $\delta_1, \delta_2, \delta_3$ *such that*

$$\begin{aligned} K_2 \subset \mathbf{\Sigma}_2 : (-\delta_3, \delta_3) \subset K_1 \cdot (-\delta_2, \delta_2) \subset \\ \mathbf{\Sigma}_1 : (-\delta_1, \delta_1) \subset V, \end{aligned} \tag{1}$$

and the second, third, and forth sets are collars (see Fig. 1).

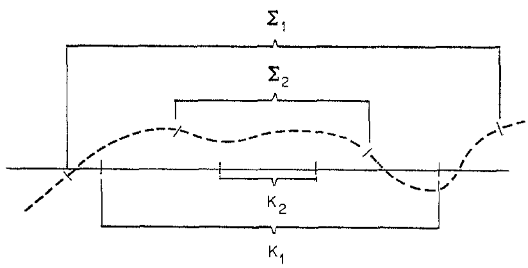

FIGURE 1

Proof. Let $V = \Sigma \cdot (-5\delta, 5\delta)$, and let $m \subset u \subset \Sigma$ be such that m is compact, u is open (rel Σ), and $K_1 \subset \mathrm{int}(m)(\mathrm{rel}\ \Sigma)$. Choose $\theta \in (0, \delta/2)$ and let $U = u \cdot (-5\delta + \theta/2, 5\delta - \theta/2)$, so that U is open and $\mathrm{Cl}(U) \subset V$. (Think of f as the flow generated in the plane by $\dot{x} = 0$, $\dot{y} = -1$, of Σ as $\{(x, y) \mid 0 < x < 1, y = 0\}$, and of V as $\{(x, y) \mid 0 < x < 1, |y| < 5\}$. Then U is a slightly smaller open rectangle sitting inside V.) Also, set $M = m \cdot [-\delta/2, \delta/2]$. Choose any open neighborhood $\Phi \subset F$ of f such that if $f' \in \Phi$, then

$$M : [-4\delta, 4\delta] \subset U. \tag{2}$$

Next choose relatively open neighborhoods w_i of K_i, $i = 1, 2$, such that $K_2 \subset w_2 \subset \mathrm{Cl}(w_2) \subset \mathrm{int}(K_1) \subset w_1 \subset \mathrm{Cl}(w_1) \subset \mathrm{int}(m)$, where the interiors are rel Σ. For $i = 1, 2$ let $W_i = w_i \cdot (-5\delta + i\theta, 5\delta - i\theta)$, so $W_2 \subset \mathrm{Cl}(W_2) \subset W_1 \subset \mathrm{Cl}(W_1) \subset U$. Modify Φ by requiring that if $f' \in \Phi$, then

$$K_1 : [-3\delta, 3\delta] \subset W_1, \tag{3}$$

$$K_2 : [-\delta, \delta] \subset W_2, \text{ and} \tag{4}$$

$$(M \cap W_2) : [-\delta, \delta] \subset \mathrm{int}(K_1) \cdot (-2\delta, 2\delta). \tag{5}$$

Finally, for $p \in V$ let

$$g(p) = (-1)(\pi_2 \circ \varphi_{5\delta}^{-1})(p), \qquad \text{where} \qquad \varphi_{5\delta} : \Sigma \times (-5\delta, 5\delta) \to X$$

is as in Section 3, and $\pi_2 : \Sigma \times (-5\delta, 5\delta) \to (-5\delta, 5\delta)$ is the obvious projection. Then $g : V \to R$ is continuous, and f is gradient-like on V w.r.t. g. By Lemma 5.2 we can further restrict Φ, still leaving this set open and containing f, by requiring that each $f' \in \Phi$ be gradient-like on U w.r.t. a continuous function $g' : U \to R$, where for each $p \in U$, $| g'(p) - g(p)| < \delta/4$. Then notice that for $q \in \Sigma$ we have $g(q \cdot s) = -s$, so that

$$| g'(q : s) - (-s)| < \delta/2, \qquad q \in \Sigma \cap U, \qquad | s | \leqslant 4\delta, \tag{6}$$

and, in particular,

$$(g')^{-1}(\{0\}) \subset \Sigma \cdot (-\delta/2, \delta/2). \tag{7}$$

Now choose $f' \in \Phi$, and for $i = 1, 2$ let $\mathbf{\Sigma}_i = (g')^{-1}(\{0\}) \cap W_i$. These are local sections by Proposition 3.2, and obviously $\mathbf{\Sigma}_2 \subset \mathbf{\Sigma}_1$. Since $w_i \subset m$, (7) shows that $\mathbf{\Sigma}_i \subset M$, $i = 1, 2$. If $p \in K_2$, then (4) implies that $p : [-\delta, \delta] \subset W_2 \subset U$, and by (6) we have $g'(p : \delta) < -\delta/2, g'(p : -\delta) > \delta/2$. Thus there is exactly one point of $p : [-\delta, \delta]$ which lies in $\mathbf{\Sigma}_2$. But then $K_2 \subset \mathbf{\Sigma}_2 : (-\delta, \delta)$, and since $\mathbf{\Sigma}_2 \subset M$, this is a collar of $\mathbf{\Sigma}_2$ by (2) and the last statement of Proposition 3.2. The first inclusion of (1) follows by letting $\delta_3 = \delta$. The second inclusion follows from (5) by letting $\delta_2 = 2\delta$. To obtain the third inclusion, argue as for the first, using (3) in place of (4), to see that $K_1 \subset \mathbf{\Sigma}_1 : (-3\delta, 3\delta)$. By (6) we also have $K_1 \cdot (-2\delta, 2\delta) \subset \mathbf{\Sigma}_1 : (-3\delta, 3\delta)$, and the third inclusion follows if we let $\delta_1 = 3\delta$. The forth inclusion is immediate from (2), since $U \subset V$. Q.E.D.

Let B be a block, and let $\sigma : B \to [0, \infty]$ be the continuous function introduced in the proof of Lemma 4.3, i.e., $\sigma(p) = \inf\{| t | \mid p \cdot t \in b^+ \cup b^-\}$, $p \in B$. Let $\theta \in (0, \infty)$ be such that $\sigma \mid \partial B \leqslant \theta$, and let $B_1 = \mathrm{Cl}(B - D)$, where $D = \{p \in B \mid \sigma \mid \mathcal{O}(p, B) \leqslant \theta\}$. It is easy to check that B_1 is also a block, and that $A_1^\pm = A^\pm$, $A_1 = A$, $a_1^\pm = a^\pm$, and $S_1 = S_1(B_1) = S(B)$. B_1 is called a *shave* of B.

LEMMA 5.4. *Let S be an isolated invariant set, and let B_1 and B_2 be blocks isolating S, with B_2 a shave of B_1. Let $V_1^\pm$ be disjoint collars of* int $b_1^\pm$(rel $\Sigma^\pm$), *and set $V_1 = V^+ \cup V^-$. Then there is an open neighborhood $\Phi \subset F$ of f such that each $f' \in \Phi$ admits two blocks $\mathbf{B}_1$ and $\mathbf{B}_2$ with the following properties:*

(a) *There are collars $\mathbf{V}_1^\pm$ of $\mathbf{b}_1^\pm$* (w.r.t. f') *and collars $V_2^\pm$ of $b_2^\pm$* (w.r.t. f) *such that for $\mathbf{V}_1 = \mathbf{V}_1^+ \cup \mathbf{V}_1^-$ and $V_2 = V_2^+ \cup V_2^-$ we have the inclusions*

$$\begin{aligned}(\mathbf{B}_2, \mathbf{b}_2^+) &\subset (B_2 \cup V_2, b_2^+ \cup V_2^+) \subset (\mathbf{B}_1 \cup \mathbf{V}_1, \mathbf{b}_1^+ \cup \mathbf{V}_1^+)\\ &\subset (B_1 \cup V_1, b_1^+ \cup V_1^+),\end{aligned} \tag{8}$$

and

(b) *$\mathbf{B}_1$ and $\mathbf{B}_2$ isolate the same invariant set* w.r.t. f'.

Proof. Assume that $B_2 = \mathrm{Cl}(B_1 - \{p \in B_1 \mid \sigma_1 \mid \mathcal{O}(p, B_1) \leqslant \theta\})$, where $\theta > \sigma_1 \mid \partial B_1 \geqslant 0$, and set $B_3 = \mathrm{Cl}(B_1 - \{p \in B_1 \mid \sigma_1 \mid \mathcal{O}(p, B_1) \leqslant 2\theta)$, so that B_3 is a second shave of B_1. Choose $\lambda \in (0, \theta/2)$ so small that $\sigma_1 \mid \partial B_1 < \theta - \lambda$, and for $i = 1, 2$ let $K_i^{\pm} = \{p \in b_i^{\pm} \mid \exists q \in \mathcal{O}(p, B_1)$ such that $\sigma_1(q) \geqslant i\theta - \lambda\}$. For the same i let

$$\epsilon_i^{\pm} = \min d(\mathcal{O}(b_i^{\pm} - \mathrm{int}(K_i^{\pm}), B_i), B_{i+1}),$$

where we agree that $d(\varnothing, B_{i+1}) = 1$, and define $\epsilon = \frac{1}{2}\min\{\epsilon_1^{\pm}, \epsilon_2^{\pm}\}$. We obviously have $b_{i+1}^{\pm} \subset \mathrm{int}(K_i^{\pm})(\mathrm{rel}\ \Sigma_1^{\pm})$, so that $\mathcal{O}(b_i^{\pm} - \mathrm{int}(K_i^{\pm}), B_i)$ and B_{i+1} are disjoint closed sets (provided the first is nonempty), and we conclude that $\epsilon > 0$ (see Fig. 2).

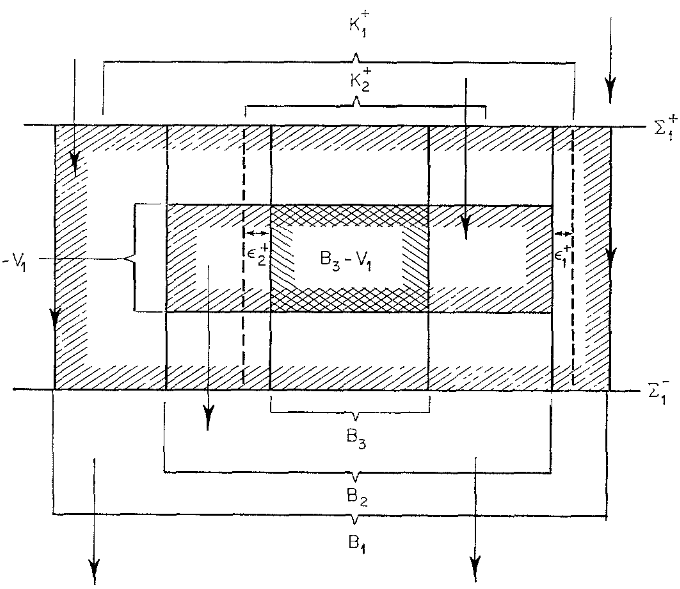

FIGURE 2

Since $V_1^{\pm}$ is a collar of $\mathrm{int}(b_1^{+})(\mathrm{rel}\ \Sigma_1^{+})$, we must have

$$V_1^{+} = \mathrm{int}(b_1^{+}) \cdot (-\delta_1, \delta_1) \qquad \text{for some} \quad \delta_1 > 0.$$

We may also assume that $V_1^{-} = b_1^{-} \cdot (-\delta_1, \delta_1)$. Choose $\mu > 0$ so small that if $p \in (K_1^{\pm} \cdot \mp\delta/2)$ and $d(p, q) < \mu$, then $q \in (V_1^{\pm} - B_1)$. Also, choose $t_0 > 0$ so large that for all $p \in (B_1 - (B_3 - B_1))$ we have $p \cdot [0, t_0] \not\subset (B_1 \cup V_1 - (B_3 - V_1))$ and $p \cdot [-t_0, 0] \not\subset (B_1 \cup V_1 - (B_3 - V_1))$. Let Φ' be the set of all $f' \in F$ such that

$$d(p : t, p \cdot t) < \gamma = \min\{\mu, \epsilon\}, \qquad p \in \mathrm{Cl}(B_1 \cup V_1), \qquad |\, t \,| \leqslant t_0. \tag{9}$$

For $i = 1, 2$ let $M_i = \{p \in B_1 \mid d(p, B_{i+1}) \leqslant \gamma\}$. By the choice of μ, (9) guarantees that any $f' \in \Phi'$ will carry any $p \in (M_1 - (B_3 - V_1^+))$ out of B_1 in negative time greater than $-t_0$, and will carry any $p \in (M_1 - (B_3 - V_1^-))$ out of B_1 in positive time less than t_0. By the choice of ϵ, for all $f' \in \Phi'$ we have $p : [-t_0, 0] \cap K_i^+ \neq \varnothing$ when $p \in (M_i - (B_3 - V_1^+))$, and $p : [0, t_0] \cap K_i^- \neq \varnothing$ when $p \in (M_i - (B_3 - V_1^-))$, $i = 1, 2$.

By Lemma 5.3 there is an open neighborhood $\Phi'' \subset F$ of f such that each $f' \in \Phi''$ admits four local sections $\mathbf{\Sigma}_1^\pm$, $\mathbf{\Sigma}_2^\pm$ such that

$$K_2^\pm \subset \mathbf{\Sigma}_2^\pm : (-\boldsymbol{\delta}_2, \boldsymbol{\delta}_2); \tag{10}$$

$$\mathbf{\Sigma}_2^\pm : (-\boldsymbol{\delta}_2, \boldsymbol{\delta}_2) \subset K_1^\pm \cdot (-\delta_1, \delta_1); \tag{11}$$

$$K_1^\pm \cdot (-\delta_1, \delta_1) \subset \mathbf{\Sigma}_1^\pm : (-\boldsymbol{\delta}_1, \boldsymbol{\delta}_1); \text{ and} \tag{12}$$

$$\mathbf{\Sigma}_1^\pm : (-\boldsymbol{\delta}_1, \boldsymbol{\delta}_1) \subset V_1^\pm. \tag{13}$$

Here δ_1, $\boldsymbol{\delta}_1$, and $\boldsymbol{\delta}_2$ are positive constants, and all sets except $K_2^\pm$ are collars.

Let $\Phi = \Phi' \cap \Phi''$, and for $f' \in \Phi$ and $p \in (M_1 - V_1)$ define

$$\boldsymbol{\sigma}^+(p) = \inf\{t \leqslant 0 \mid p : [t, 0] \cap \mathbf{\Sigma}_1^+ = \varnothing\},$$

$\boldsymbol{\sigma}^-(p) = \sup\{t \geqslant 0 \mid p : [0, t] \cap \mathbf{\Sigma}_1^- = \varnothing\}$. Observe that if $|\boldsymbol{\sigma}^\pm(p)| < \infty$, then $p : \boldsymbol{\sigma}^\pm(p) \in \mathbf{\Sigma}_1^\pm$. Also, (12) and the remarks following (9) show that for any $p \in (M_1 - V_1)$ we have

$$\left.\begin{aligned} &p : (\boldsymbol{\sigma}^+(p), 0] \cap (\mathrm{Cl}(\mathbf{\Sigma}_1^+) - \mathbf{\Sigma}_1^+) = \varnothing, \\ \text{and}\quad &p : [0, \boldsymbol{\sigma}^-(p)) \cap (\mathrm{Cl}(\mathbf{\Sigma}_1^-) - \mathbf{\Sigma}_1^-) = \varnothing. \end{aligned}\right\} \tag{14}$$

It follows from (14) and Proposition 3.1 that $\boldsymbol{\sigma}^+ : (M_1 - V_1) \to [-\infty, 0]$ and $\sigma^- : (M_1 - V_1) \to [0, \infty]$ are continuous.

For $i = 1, 2$ let $D_i^+ = \bigcup p : (\boldsymbol{\sigma}^+(p), 0]$, $D_i^- = \bigcup p : [0, \boldsymbol{\sigma}^-(p))$, where the unions are taken over all $p \in (B_{i+1} - V_1)$, and let $D_i = D_i^+ \cup D_i^-$. With $\mathbf{\Sigma}_1^\pm$ and $\mathbf{\Sigma}_2^\pm$ as the associated local sections, we assert that $\mathbf{B}_1 = \mathrm{Cl}(D_1)$ and $\mathbf{B}_2 = \mathrm{Cl}(D_2)$ are blocks which satisfy the requirements of the lemma. First we show that $\mathbf{B}_1$ is a block. The proof that $\mathbf{B}_2$ is a block is similar.

First we observe from (9) that

$$\mathbf{B}_1 \subset \mathcal{O}(M_1, B_1 \cup V_1). \tag{15}$$

In combination with (14), this shows that $\mathbf{B}_1$ satisfies Condition 3.3(a). Next observe that $(\partial\mathbf{B}_1 - (\mathbf{\Sigma}_1^+ \cup \mathbf{\Sigma}_1^-)) \subset \mathcal{O}(B_1 - B_3, B_1 \cup V_1)$, and so the comments following (9) show that

$$|\boldsymbol{\sigma}^\pm(p)| < \infty \qquad \text{if} \quad p \in (\partial\mathbf{B}_1 - (\mathbf{\Sigma}_1^+ \cup \mathbf{\Sigma}_1^-)).$$

Using the continuity of $\sigma^{\pm}$, it is a simple matter to see that

$$p : [\sigma^+(p), \sigma^-(p)] \subset \partial \mathbf{B}_1 ,$$

and since $p : \sigma^+(p) \in \boldsymbol{\Sigma}_1{}^+$, $p : \sigma^-(p) \in \boldsymbol{\Sigma}_1{}^-$, we have verified Condition 3.3(c). Condition 3.3(b) follows easily from the definitions of $D_1{}^{\pm}$.

Now to the inclusions in (8). Since $\mathbf{b}_2{}^+ \subset \boldsymbol{\Sigma}_2{}^+$, we see from (11) that $\mathbf{b}_2{}^+ \subset K_1{}^+ \cdot (-\delta_1 , \delta_1)$. However, by the choice of ϵ, (9) guarantees that $(K_1{}^+ - b_2{}^+) \cdot (-\delta_1 , \delta_1) \cap \mathbf{b}_2{}^+ = \varnothing$, so that we actually have $\mathbf{b}_2{}^+ \subset b_2{}^+ \cdot (-\delta_1 , \delta_1)$. With the observation that $\mathbf{B}_2 \subset \mathcal{O}(M_2 , B_1 \cup V_1) \subset B_1 \cup V_1$, the first inclusion of (8) follows. The second and third follow from similar arguments, first replacing (11) by (12), and then (12) by (13). This gives (a), and (b) is immediate from the remarks following (9). Q.E.D.

Proof of Theorem 5.1. Shave B to B_1. By Lemma 5.4(a) there is an open neighborhood $\Phi \subset F$ of f such that any $f' \in \Phi$ admits two blocks $\mathbf{B}$ and $\mathbf{B}_1$ such that

$$(\mathbf{B}_1 , \mathbf{b}_1{}^+) \subset (B_1 \cup V_1 , b_1{}^+ \cup V_1{}^+) \subset (\mathbf{B} \cup \mathbf{V}, \mathbf{b}^+ \cup \mathbf{V}^+) \subset (B \cup V, b^+ \cup V^+),$$

where $V_1{}^{\pm}$ is a collar of $b_1{}^{\pm}$ (w.r.t. f), $\mathbf{V}^{\pm}$ is a collar of $\mathbf{b}^{\pm}$ (w.r.t. f'), $V_1 = V_1{}^+ \cup V_1{}^-$, and $\mathbf{V} = \mathbf{V}^+ \cup \mathbf{V}^-$. The last inclusion gives the first statement of our theorem. To see the second, apply the functor H^* to the chain of inclusions, and recall that $(B_1 , b_1{}^+)$, $(\mathbf{B}_1 , \mathbf{b}_1{}^+)$, and (B, b^+) are strong deformation retracts of $(B_1 \cup V_1 , b_1{}^+ \cup V_1{}^+)$, $(\mathbf{B}_1 \cup \mathbf{V}_1 , \mathbf{b}_1{}^+ \cup \mathbf{V}_1{}^+)$, and $(B \cup V, b^+ \cup V^+)$, respectively. This yields a sequence of maps

$$H^*(B, b^+) \xrightarrow{\alpha} H^*(\mathbf{B}, \mathbf{b}^+) \xrightarrow{\beta} H^*(B_1 , b_1{}^+) \xrightarrow{\gamma} H^*(\mathbf{B}_1 , \mathbf{b}_1{}^+).$$

$\beta \circ \alpha$ is an isomorphism by Theorem 4.4. Lemma 5.4(b) allows us to apply the same theorem to $\gamma \circ \beta$, showing that it too is an isomorphism. We conclude that β, and then α, is an isomorphism. Q.E.D.

6. Examples

(a) A nonempty invariant set I is *positively attracting* if there is an open neighborhood W of I such that $\mathcal{O}_\omega(p) \subset I$ for all $p \in W$. I is *positively stable* if for each open neighborhood U of I there is an open neighborhood $V \subset U$ of I such that $V \cdot [0, \infty) \subset U$. I is *positively asymptotically stable* if it is both positively attracting and positively stable. Negative asymptotic stability is defined similarly, using $\mathcal{O}_\alpha(p)$ in place of $\mathcal{O}_\omega(p)$, and $(-\infty, 0]$ in place of $[0, \infty)$. Note that positively and negatively asymptotically stable invariant sets are isolated.

THEOREM 6.1. *Let B be an isolating block. Then $S = S(B)$ is positively asymptotically stable if and only if $a^- = \varnothing$.*

Proof. First we claim that if $a^- = \varnothing$, then $A \cap \mathrm{Cl}(B - A) = \varnothing$. Otherwise there is a point $p \in A$ and a sequence $\{p_n\} \subset B - A$ such that $p_n \to p$. If $q_n = \mathcal{O}(p_n, B) \cap b^-$, $n = 1, 2,\ldots$, and if q is a limit point of $\{p_n\}$, then one easily checks that $q \in a^-$, a contradiction.

Now assume that $a^- = \varnothing$, and let Y be a neighborhood of $A = A^+$ such that $\mathrm{Cl}(B - A) \cap Y = \varnothing$. Then $W = Y \cap \mathrm{int}(B)$ is such that $p \in W$ implies $\mathcal{O}_\omega(p) \subset I$, showing that I is positively attracting. If U is any neighborhood of I (assume $U \subset \mathrm{int}(B)$ to avoid a trivial case), then by Theorem 3.4 we can find a block $B_1 \subset U$ which also isolates I. Again we have $a_1{}^- = \varnothing$ (Lemma 4.1) and we can choose a neighborhood Y' of $A_1 = A_1{}^+$ such that $\mathrm{Cl}(B_1 - A_1) \cap Y' = \varnothing$. Then for $V = \mathrm{int}(B_1) \cap Y'$ we have $V \cdot [0, \infty) \subset U$, showing that I is positively stable.

The converse is obvious. Q.E.D.

Similarly, $S = S(B)$ is negatively asymptotically stable if and only if $a^+ = \varnothing$.

THEOREM 6.2. [27, 4]. *If $I \subset X$ is a nonempty invariant set, then one of the following holds:*

(1) *I is not isolated;*

(2) *I is positively asymptotically stable;*

(3) *I is negatively asymptotically stable;*

(4) *There are points $p, q \notin I$ such that $\mathcal{O}_\alpha(p) \subset I$ and $\mathcal{O}_\omega(q) \subset I$.*

Our proof assumes that X is compact. The theorem also holds for X locally compact, provided I is compact.

Proof. If (1) does not hold we can apply Theorem 3.4 to isolate I with a block B. From the previous theorem we see that (2) holds if $a^- = \varnothing$. Similarly, (3) holds if $a^+ = \varnothing$. If a^- and a^+ are nonempty, choose p in the first set and q in the second, and (4) holds. Q.E.D.

(b) Let B be an isolating block, let $S = S(B)$, and let

$$\cdots \to H^{q-1}(a^+) \to H^q(B, b^+) \to H^q(S) \to H^q(a^+) \to \cdots$$

be the sequence of S, computed using B. If $H^q(B, b^+) \neq 0$, then either $H^{q-1}(a^+) \neq 0$ or else $H^q(S) \neq 0$. In the first case there must be a point $p \in a^+$, and thus $S \neq \varnothing$, since $\varnothing \neq \mathcal{O}_\omega(p) \subset S$. But obviously $S \neq \varnothing$ in the second case also, so if $H^q(B, b^+) \neq 0$ for some $q \geqslant 0$, then $S \neq \varnothing$. Considering this remark along with Theorem 5.1, we see that $H^q(B, b^+) \neq 0$

for some $q \geqslant 0$ implies that there are nontrivial isolated invariant sets close to S under small perturbations of f.

As an example, we can obtain a block B by rotating Figure 3 about the y-axis. We see that $H^*(B, b^+) \neq 0$, and conclude that there is a nontrivial invariant set inside B. This particular block arises quite naturally in the study of the second collinear Lagrange point of the restricted three-body problem [6, 16].

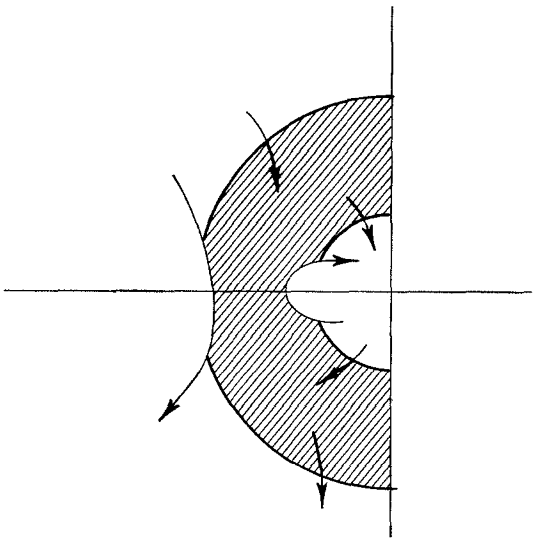

FIGURE 3

(c) Let $B \subset R \times R \times R$ be a closed two-cell crossed with $[0, 1]$, with b^+ the top and b^- the bottom. Assume that there is a knotted orbit in B which runs from b^+ to b^-, and let α be any arc in b^+ which runs from this orbit to the boundary. If the flow carries α to b^-, then a disc is swept out which is bounded by a knotted S^1. Since this is impossible, some point of α is in a^+, and we conclude that $S \neq \varnothing$, even though $H^*(B, b^+) = 0$. Notice that both the knotted orbit and the argument persist under small perturbations.

(d) In Section 4 we introduced the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^q(B, b^+) & \longrightarrow & H^q(B) & \longrightarrow & H^q(b^+) & \longrightarrow & H^{q+1}(B, b^+) & \longrightarrow \cdots \\
& \approx \big\downarrow i^* & & \big\downarrow j_1^* & & \big\downarrow j_2^* & & \approx \big\downarrow i^* & \\
\cdots \longrightarrow & H^q(A, a^+) & \longrightarrow & H^q(S) & \longrightarrow & H^q(a^+) & \longrightarrow & H^{q+1}(A, a^+) & \longrightarrow \cdots
\end{array}$$

of a block B. Recall that the vertical maps were induced by inclusions. From the 5–lemma we see that $j_1^* : H^*(B) \to H^*(S)$ is an isomorphism if and only if $j_2^* : H^*(b^+) \to H^*(a^+)$ is an isomorphism.

Suppose S is a periodic orbit in $R \times R \times R$, that B is a solid torus isolating S, and that S is a strong deformation retract of B. Then $j_1^* : H^*(B) \to H^*(S)$ is an isomorphism, and we conclude that b^+ and a^+ must have the same cohomology groups. Easy application: If b^+ is an annulus, then $a^+ \subset b^+$ cannot be a disc.

In Fig. 4 we have a situation where $j_2^* : H^*(b^+) \to H^*(a^+)$ is an isomorphism, and we conclude that S is a Čech circle, i.e., $H^*(S) \approx H^*(S^1)$.

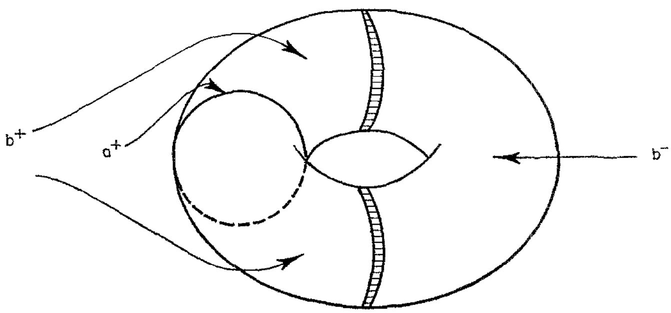

FIGURE 4

(e) Let B be an n-ball, $n > 2$, and let b^+ consist of a closed $(n-1)$-cell at the north pole of B and k disjoint copies of $S^{n-2} \times [0, 1]$ paralleling the equator of B. In the long exact sequence of (B, b^+) we have $H^{n-2}(b^+) \approx H^{n-1}(B, b^+)$, and obviously $H^{n-2}(b^+)$ has k generators. From the diagram of B we conclude that $H^{n-2}(a^+)$ has at least k generators. In particular, $H^{n-2}(a^+) \neq 0$ if $k \geqslant 1$.

(f) Let S be an isolated invariant set. By replacing "+" superscripts by "−" superscripts in the arguments of Section 4, we find that the algebra $H^*(B, b^-)$ and the sequence

$$\cdots \to H^{q-1}(a^-) \to H^q(B, b^-) \to H^q(S) \to H^q(a^-) \to \cdots$$

are additional invariants of S. Using this sequence in combination with the first theorem of (a), we conclude

THEOREM 6.3. *S is positively asymptotically stable if and only if the inclusion $S \subset (B, b^-)$ induces an isomorphism $H^*(B, b^-) \approx H^*(S)$.*

Now let B be an isolating block with $S = S(B)$ positively asymptotically stable, and consider the diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^{q-1}(b^-) \arrow[r, "\delta"] & H^q(B, b^-) \arrow[r, "k^*"] \arrow[d, "i^*"'] & H^q(B) \arrow[r, "l^*"] \arrow[dl, "j^*"] & H^q(b^-) \arrow[r] & \cdots \\ & & H^q(S) & & & \end{tikzcd}$$

where i, j and k are inclusions and the top row is the long exact sequence of (B, b^-). As $i^* = j^* \circ k^*$, and as i^* is an isomorphism, k^* must be an injection. By letting $\boldsymbol{\delta} = i^* \circ \delta$ and $\mathbf{k}^* = k^* \circ i^*$, we obtain the following result.

THEOREM 6.4. *Let B be a block for which $S = S(B)$ is positively asymptotically stable. Then there is a long exact sequence*

$$\cdots \longrightarrow H^{q-1}(b^-) \xrightarrow{\boldsymbol{\delta}} H^q(S) \xrightarrow{\mathbf{k}^*} H^q(B) \xrightarrow{l^*} H^q(b^-) \longrightarrow \cdots$$

such that $\mathbf{k}^*$ is an injection, l^* is a surjection, and $\boldsymbol{\delta}$ is the zero map.

In the case of a smooth flow, one can always choose an isolating block B in the form of a smooth manifold with boundary [11, 32]. If $S = S(B)$ is positively asymptotically stable, then by Theorem 6.4 there is an injection $k^* : H^*(S) \to H^*(B)$. We conclude that $H^*(S)$ is finitely generated. Easy application: A p-adic solenoid cannot be a positively asymptotically stable invariant set for a smooth flow. Interestingly enough, a p-adic solenoid can be a positively asymptotically stable invariant set for a smooth discrete flow, as is shown by an example of Williams [31].

As a second application of Theorem 6.4, suppose $B \subset R \times R \times R$ is a 3-ball, and $I \subset \operatorname{int}(B)$ is a periodic orbit. Then $H^1(I)$ cannot inject into $H^1(B)$, so that if I is positively asymptotically stable, it cannot be the maximal invariant set inside B. Notice that no assumptions are made on b^+ or a^+.

(g) Suppose S is positively asymptotically stable and B is a block isolating S. By the methods of Section 5 we see that if f is perturbed slightly to f', then f' admits a block $\mathbf{B}$ near B such that $H^*(\mathbf{B}, \mathbf{b}^-) \approx H^*(B, b^-)$. But it is clear that for f' sufficiently close to f we must have $\mathbf{S} = \mathbf{S}(\mathbf{B})$ positively asymptotically stable, and from Theorem 6.3 we conclude that $H^*(\mathbf{S}) \approx H^*(S)$. It is easy to construct examples for which $\mathbf{S}$ and S are not homeomorphic.

REFERENCES

1. J. AUSLANDER AND P. SIEBERT, Prolongations and generalized Liapunov functions, *in* "Proceedings of the International Symposium Nonlinear Differential Equations and Nonlinear Mechanics," Colorado Springs, 1961, Academic Press, New York, 1963.
2. J. AUSLANDER AND P. SIEBERT, Prolongations and stability in dynamical systems, *Ann. Inst. Fourier Grenoble* **14** (1964), 237–268.
3. N. P. BHATIA, On asymptotic stability in abstract dynamical systems, *Math. Systems Theory*, **1** (1967), 113–127.
4. N. P. BHATIA, Semidynamical flow near a compact invariant set, *in* 'Topological Dynamics" (J. Auslander and W. Gottschalk, Eds.), pp. 81–96, Benjamin, New York, 1968.

5. C. CONLEY, Invariant sets in a monkey saddle, *in* "Proceedings of the U.S.-Japan Seminar Differential and Functional Equations," pp. 443–447, Benjamin, New York, 1967.
6. C. CONLEY, On the ultimate behavior of orbits with respect to an unstable critical point, *J. Differential Equations* **5** (1969), 136–158.
7. C. CONLEY, Invariant sets which carry a one-form, *J. Differential Equations* 8 (1970), 587–594.
8. C. CONLEY, On the continuation of invariant sets of a flow, *in* "Actes, Congres Intern. Math., 1970," Vol. 2, pp. 909–913, Gauthier–Villars, Paris, 1971.
9. C. CONLEY, On a generalization of the Morse index, to appear.
10. C. CONLEY AND R. EASTON, On isolated invariant sets and isolating submanifolds, *in* "Advances in Differential and Integral Equations" (J. Nohel, Ed.), pp. 97–104, SIAM, Philadelphia, 1969.
11. C. CONLEY AND R. EASTON, Isolated invariant sets and isolating blocks, *Trans. Amer. Math Soc.* **158** (1971), 35–61.
12. R. EASTON, On the existence of invariant sets inside a submanifold convex to a flow, *J. Differential Equations* **7** (1970), 54–68.
13. R. EASTON, Locating invariant sets, *in* "Proceedings of the Symposium on Pure Mathematics," Vol. 14, pp. 55–59, Amer. Math. Soc., Providence, RI, 1970.
14. R. EASTON, Regularization of vector fields by surgery, *J. Differential Equations* **10** (1971), 92–99.
15. P. HARTMAN, "Ordinary Differential Equations," Wiley, New York, 1964.
16. R. MCGEHEE, Some Homoclinic Orbits for the Restricted Three-Body Problem, thesis, University of Wisconsin, 1969.
17. R. MCGEHEE, Parabolic orbits in the three-body problem, to appear.
18. J. T. MONTGOMERY, Cohomology of isolated invariant sets under perturbation, to appear.
19. V. V. NEMYTSKII AND V. V. STEPANOV, "Qualitative Theory of Differential Equations," Princeton University Press, Princeton, NJ, 1960.
20. D. ROD, Invariant sets in the monkey saddle, to appear.
21. J. SMOLLER AND C. CONLEY, Viscosity matrices for two-dimensional non-linear hyperbolic systems, *Comm. Pure Appl. Math.* **23** (1970), 867–884.
22. J. SMOLLER AND C. CONLEY, Viscosity matrices for two-dimensional non-linear hyperbolic systems II, to appear.
23. J. SMOLLER AND C. CONLEY, Shock waves as limits of progressive wave solutions of higher order equations, *Comm. Pure Appl. Math.*, to appear.
24. J. SMOLLER AND C. CONLEY, Shock waves as limits of progressive wave solutions of higher order equations II, to appear.
25. E. THOMAS, One-dimensional minimal sets, to appear.
26. TARO URA, On the flow outside a closed invariant set, *Contrib. Differential Equations* **3** (1964), 249–294.
27. TARO URA AND IKUO KIMURA, Sur le Courant Extérieur à une Région Invariante; Théorème de Bendixon, *Comment. Math. Univ. Sancti Pauli* **8** (1960), 23–29.
28. T. WAŻEWSKI, Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires, *Ann. Soc. Polon. Math.* **20** (1947), 279–313.
29. T. WAŻEWSKI, Sur une méthode topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles, *in* "Proceedings of the International Congress of Mathematicians," Vol III, Noordhoff, Groningen, 1954.
30. H. WHITNEY, Regular families of curves, *Ann. of Math.* **34** (1933), 244–270.

31. R. F. Williams, Non-compact Lie group actions, *in* "Proceedings of the Conference on Transformation Groups," New Orleans, 1967 Springer, Berlin, 1968.
32. F. W. Wilson, Jr. and J. A. Yorke, Lyapunov functions and isolating blocks, *J. Differential Equations*, to appear.
33. V. I. Zubov, "Methods of A. M. Liapunov and their Application," Noordhoff, Groningen, 1964.